

Partial Fractions

It is easy to combine two fractions into one such as

$$\frac{1}{x+3} + \frac{2}{x-4} = \frac{3x+2}{(x+3)(x-4)}.$$

The method of partial fractions is about the reverse process. We want to find constants A and B such that

$$\frac{3x+2}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}.$$

To decompose a rational function into partial fractions, we proceed as follows.

1. If the degree of the numerator $N(x)$ is greater than or equal to the degree of the denominator $D(x)$, use long division to obtain

$$\frac{N(x)}{D(x)} = P(x) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Apply the next steps to $\frac{N_1(x)}{D(x)}$.

2. Completely factor the denominator $D(x)$ into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where ax^2+bx+c is an *irreducible quadratic*.

3. For *each* factor of the form $(px+q)^m$, the partial fractions decomposition must include the following m terms.

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. For *each* factor of the form $(ax^2+bx+c)^n$, the partial fractions decomposition must include the following n terms.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

5. Find the values of all the constants.

When the denominator has distinct roots, a useful trick to find the constants in a partial fractions decomposition is the cover-up method. Let's illustrate it with an example.

Consider the following partial fractions decomposition.

$$\frac{x^2 + 5x + 4}{(x - 1)(x - 3)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x - 3} + \frac{C}{x + 2} \quad (*)$$

To find A , we could use the following two steps.

1. Multiply both sides of equation $(*)$ by $(x - 1)(x - 3)(x + 2)$ to get

$$x^2 + 5x + 4 = A(x - 3)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x - 3).$$

2. Set $x = 1$ and solve for A .

$$1 + 5 + 4 = A(-2)(3) + 0 + 0 \implies A = -\frac{5}{3}$$

The cover-up method combines these two steps in a single one as follows. To find A , we cover up $(x - 1)$ and set $x = 1$ in the left-hand side of $(*)$.

$$A = \frac{x^2 + 5x + 4}{\cancel{(x - 1)}(x - 3)(x + 2)} \Big|_{x=1} = -\frac{5}{3}$$

We can find constants B and C in the same way. To find B , we cover up $(x - 3)$ and set $x = 3$.

$$B = \frac{x^2 + 5x + 4}{(x - 1)\cancel{(x - 3)}(x + 2)} \Big|_{x=3} = \frac{14}{5}$$

To find C , we cover up $(x + 2)$ and set $x = -2$.

$$C = \frac{x^2 + 5x + 4}{(x - 1)(x - 3)\cancel{(x + 2)}} \Big|_{x=-2} = -\frac{2}{15}$$